Problem Set 5: due Monday 30th March

1. In the general case where $\nu \propto R^\gamma$, the similarity solution for the evolution of an isolated disc takes the form

$$\Sigma(R,t) = \frac{M_d(0)(2-\gamma)}{2\pi R_0^2 r^\gamma} \tau^{\frac{-(5/2-\gamma)}{2-\gamma}} \exp\left(-\frac{r^{2-\gamma}}{\tau}\right) , \quad (1)$$

where $r = R/R_0$, $\tau = t/t_\nu + 1$ and $t_\nu = R_0^2/(3(2-\gamma)^2\nu_0)$.

a) Show that this solution satisfies the diffusion equation for the evolution of disc surface density.

b) Sketch (or plot) the form of $\Sigma(R,t)$ for $\gamma = 1$ at $\tau = 1$, 3, 10, 30, 100, and identify its important features. Repeat this exercise for $\gamma = 3/2$.

c) Find expressions for the disc mass, and the accretion rate as $R \to 0$, as functions of time. Show that these reduce to power-laws in the limit $t \gg t_\nu$.

2. **a)** A typical T Tauri star has mass $M_* = 0.5M_\odot$, radius $R_* = 2R_\odot$, and an accretion rate of $3 \times 10^{-8}\mathrm{M_\odot yr^{-1}}$. If the stellar magnetosphere truncates the disc at a radius $5R_*$, and gas from this point falls freely along the field lines to the stellar surface, what is the accretion luminosity?

b) Assume that the accretion shock covers a fraction $f = 0.5\%$ of the stellar surface, and that the accretion shock radiates away all of the accretion luminosity. If the shock radiates as a black body, what is its temperature? Explain why the accretion shock gives rise to a characteristic "UV excess".

c) The lowest detectable accretion rates on to solar-mass T Tauri stars are $\sim 10^{-10}\mathrm{M_\odot yr^{-1}}$, because at lower values it becomes impossible to detect the accretion luminosity against the "background" stellar luminosity. If the detection threshold for accretion luminosity can be thought of as a fixed fraction of the stellar luminosity, estimate the lowest detectable accretion rate on to a $0.01\mathrm{M_\odot}$ brown dwarf. (Assume that $L_* \propto M_*^{1.5}$, and that $R_* \propto M_*^{0.5}$.)

d) In practice we can measure brown dwarf accretion rates down to around an order of magnitude below this estimate: suggest possible reasons for this over-estimate.

3. **a)** In the case of an isothermal self-gravitating disc where the disc completely dominates the potential, the vertical structure of the disc is

$$\rho(z) = \rho_0 \frac{1}{\cosh^2(z/H_{sg})} , \quad (2)$$

where the scale-height $H_{sg} = c_s^2/\pi G\Sigma$. Show that H_{sg} is approximately equal to the pressure scale-height of the disc when the disc is marginally unstable to its own gravity.

b) Formally, the dispersion relation $\omega(c_s, k)$ for axisymmetric modes in a thin, gravitationally unstable disc is

$$\omega^2 = \kappa^2 - 2\pi G\Sigma|k| + k^2 c_s^2 , \quad (3)$$

where κ is the epicyclic frequency of the disc and k is the mode wavenumber. (See Pringle & King, Chapter 12, for a derivation.) In the special case of a Keplerian potential, show that the most unstable modes have a wavelength comparable to the pressure scale-height of the disc. Explain why gravitational instabilities in thin discs are therefore expected to behave in a "local" manner.